L12. Mechanics of Nanostructures: Mechanical Resonance

Outline

1. Theory
2. Mechanical Resonance Experiments
3. Ruoff group work
   - SiO₂ nanowires
   - Quartz Fibers
   - Crystalline Boron Nanowires
4. Summary
Part One: Mechanical Resonance Method

Simple Beam Theory

The mechanical resonance testing of nanostructures has to date been based on simple beam theory.

Assumptions on geometry:
- Long and thin (L >> b, h)
- Loading is in Z direction (no axial load)

Assumption on deformation:
- Plane sections remain plane and perpendicular to the mid-plane after deformation

Natural Frequency

\[
\beta_n^2 = \frac{E_b I}{2\pi m L^4}
\]

\[ f_n = \frac{\beta_n^2}{2\pi} \sqrt{\frac{E_b I}{mL^4}} \]

\( \beta \): eigenvalue  \( E_b \): elastic modulus
\( I \): moment of inertia  \( L \): beam length
\( m \): mass per unit length

Resonance Test Principle

According to simple beam theory, the natural frequency of a cantilevered circular cross-section beam is given by:

**Natural Frequency**

\[
f_n = \frac{\beta_n^2 D}{2\pi L^2} \sqrt{\frac{E_p}{16\rho}}
\]

\(\beta_n\): constant  \(\beta_0 = 1.875\)

\(D\): diameter  \(\beta_1 = 4.694\)

\(L\): length  \(\beta_2 = 7.855\)

\(\rho\): density  \(\beta_3 = 10.996\)

**Bending Modulus:**

\[
E_b = \frac{64\pi^2 \rho L^4}{\beta_n^4 D^2 f_n^2}
\]

The bending modulus can be calculated through measurement of dimension and resonance frequency.

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Electrical Excitation

The mechanical resonance of a cantilevered structure can be excited by applying a periodic load whose frequency equals the natural frequency of the structure.

\[
F(t) = \alpha(\Delta V + V_{dc} + V_{ac}\cos\alpha t)^2
\]

\[
= \alpha(\Delta V + V_{ac})^2 + \frac{1}{2}V_{ac}^2 + 2\alpha(\Delta V + V_{ac})V_{ac}\cos\alpha t + \frac{\alpha V_{ac}^2}{2}\cos2\alpha t
\]
Mechanical Excitation

The mechanical resonance of a cantilevered structure can also be mechanically excited through the vibration of its substrate.

The working frequency range of the mechanical excitation method is here limited by the frequency response of the piezoelectric actuator.

Part Two:
Mechanical Resonance Experiments
Mechanical Resonance: Carbon Nanotube

(top) Elastic properties of nanotubes
(left) Electromechanical vibration of a MWCNT (A) thermal vibration (B) Fundamental resonance (C) First overtone resonance


Mechanical Resonance: DLC Pillar

Schematic of mechanical vibration experimental setup

SEM image of the vibration

Mechanical Resonance: ZnO Nanobelt

FIG. 1. (a) SEM image of the as-synthesized ZnO nanobelts. (b) A typical TEM image of a ZnO nanobelt and its electron diffraction pattern (inset). (c) Schematic geometrical shape of the nanobelt.


Mechanical Resonance: ZnO nanobelt (con’t)

FIG. 3. A selected ZnO nanobelt with a hooked end at (a) stationary, (b) resonance at 731 kHz in the plane almost parallel to the viewing direction, and (c) resonance at 474 kHz in the plane closely perpendicular to the viewing direction.
Mechanical Resonance: Work Function Measurement

\[ F = \beta(Q_0 + \alpha e (V_{dc} + V_{ac} \cos 2\pi ft))^2 \]
\[ = \alpha^2 \beta^2 \left( [\phi_{at} - \phi_{NBT} + eV_{dc}]^2 + e^2 V_{ac}^2 / 2 \right) + \]
\[ 2eV_{dc} (\phi_{at} - \phi_{NBT} + eV_{dc} \cos 2\pi ft + e^2 V_{ac}^2 / 2\cos 4\pi ft) \]

Figure 1. Experimental set up for measuring the work function at the tip of a ZnO nanobelt.


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Parametric Resonance

- For example: boron nanowire is driven electromechanically
- Signal includes DC and AC components
- Experimental stage is mounted inside SEM for visualization
Electromechanical Driving: Mathieu Equation

- The force due to the electrical field is similar to that on a cantilever between capacitor plates:
- The voltage-induced force on the nanowire increases as the tube bends closer to the electrode:

\[
F(x, t) \propto \frac{V^2(t)}{d^2} = \frac{V^2(t)}{(d_0 - u(x, t))^2}
\]

\[
F(x, t) \approx V^2(t) \left(q_0 + q_1 u(x, t)\right)
\]

- Equation for each mode takes the form of the damped Mathieu equation

\[
\ddot{u} + \mu \dot{u} - (a + 2e \cos t) u = f(t)
\]

Stability of the Mathieu Equation

\[
\frac{d^2 Y}{dt^2} + \mu \frac{dY}{dt} + (a + 2e \cos t)y = 0
\]

Parametric Resonance

\[
\frac{d^2 Y}{dt^2} + \mu \frac{dY}{dt} + (a + 2\varepsilon \cos t)Y = 0
\]

\[
a = \frac{1}{\Omega^2} \left( \omega_0^2 + \frac{q V_{dc}^2}{\rho A} \right)
\]

\[
\varepsilon = \frac{q V_{dc} V_{ac}}{\rho A}
\]

- Mathieu equation has unstable solutions for:
  \[a - n^2/4 = \frac{1}{4}, \frac{9}{4}, 4, \ldots\]

- Resonance is observed at driving frequencies that give the unstable values of \(a\):
  \[\Omega = \frac{2}{n} \sqrt{\omega_0^2 - q V_{dc} V_{ac}/\rho A}\]

Instability Regions for Vibrating Nanowire

- Small shifts in frequency or driving voltage can cause switch from unstable to stable behavior

- This instability can be used to sense changes in the environment of a vibrating nanowire
Part Three: Ruoff Group Work

Experiment Tool: Nanomanipulator

- Four-degree of freedom (x,y,z linear motion and rotation)
- Two separate stages (X-Y stage, Z-θ stage)
- Sub-nanometer motion resolution
Experimental Setup

(a) Electrical Excitation

(b) Mechanical Excitation

Mechanical Resonance of SiO$_2$ Nanowire

**SiO₂ Nanowire: Source**

Ultrasonically dispersed SiO₂ nanowire

TEM image (inserts: High resolution image and diffraction pattern)

Synthesized by Z.W. Pan (J.Am.Chem.Soc.’02)

Northwestern University  Rod Ruoff  Nanotechnology

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**SiO₂ Nanowire: Mechanical Resonance**

Electrical Excitation  Mechanical Excitation

Northwestern University  Rod Ruoff  Nanotechnology
SiO$_2$ Nanowire: Charge Trapping

Experiment 1

E-beam modes, time between scanlines:

- TV mode: 0.06 ms
- 3-d mode: 17 ms
- 4-th mode: 50 ms

Hitachi S4500


SiO$_2$ Nanowire: Charge Trapping

Experiment 2

\[ v_d(x) = \frac{8p_v x^3}{3\pi Ed_4} (4a - x) \]

SiO$_2$ Nanowire: Bending Modulus

\[ f_i = \frac{\beta_i^2 L}{2\pi} \frac{d}{12 \rho} \quad \beta_i = 1.875 \]

\[ y_n(x) = A_n [\sin \beta_n x - \sinh \beta_n x - B_n (\cos \beta_n x - \cosh \beta_n x)] \]

\[ A_n = \frac{\sinh \beta_n x - \sin \beta_n x}{2(\cosh \beta_n x + \cos \beta_n x)} \quad B_n = \frac{\cosh \beta_n x + \cos \beta_n x}{\sinh \beta_n x - \sin \beta_n x} \]

<table>
<thead>
<tr>
<th>Length, um</th>
<th>Diameter, nm</th>
<th>Natural frequency, kHz</th>
<th>E, GPa</th>
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<td>80 (± 5)</td>
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<td>17.3</td>
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<td>18.3</td>
<td>98</td>
<td>190.0</td>
<td>47.4 ± 8.4</td>
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</table>

The density of the SiO$_2$ is 2200 kg/m$^3$.

Mechanical Resonance of Quartz Fiber

Xinqi Chen, Sulin Zhang, Gregory J. Wagner, Weiqiang Ding, and Rodney S. Ruoff,
Mechanical resonance of quartz microfibers and boundary condition effects, Journal of Applied Physics, 95 (9), 4823-4828, 2003
Quartz Fibers

Quartz fibers were home-made by pulling a fused quartz rod (GE Quartz, Inc) on a wide flame.

**Typical sample geometry:** diameter: 30-100 μm, length: 5-10 mm

Quartz Fiber: Mechanically Induced Resonance

Optical microscope pictures of the first four modes of resonance of a quartz microfiber. The insets are the theoretical displacement curves.
Quartz Fiber: Correction for Non-uniform Diameter

Due to the pulling process, the quartz fiber diameter is not quite uniform. The resonance frequency change for a beam of circular cross-section, with linearly varying diameter, was calculated according to the following equation:

\[ \alpha = \frac{D_1 - D_0}{D_0} \]

\( \alpha \) is generally small.

\( n=0: \quad f = f_0 \times (1 - 0.42 \alpha) \)
\( n=1: \quad f = f_0 \times (1 + 0.22 \alpha) \)
\( n=2: \quad f = f_0 \times (1 + 0.42 \alpha) \)

\( f_0 \) is the corresponding resonance frequencies of a beam with uniform diameter \( D_0 \).

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Quartz Fiber: Young’s Modulus

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<tr>
<th>#</th>
<th>L, mm</th>
<th>( D_0 ), um</th>
<th>( D_1 ), um</th>
<th>( f_0 ), Hz</th>
<th>( f_1 ), Hz</th>
<th>( f_1/f_0 )</th>
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<td>65.6</td>
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</table>

Note: \( L \): length, \( D_0 \): the clamped side diameter, \( D_1 \): the free end diameter, \( f_0 \): the fundamental resonance frequency, \( f_1 \): the first overtone resonance frequency.
Mechanical Resonance of Crystalline Boron Nanowires


Boron Nanowire: Source

SEM image of boron nanowires on alumina substrate

TEM image of a boron nanowire

Boron Nanowire: Resonance

First two modes of resonance of a cantilevered BNW

Typical frequency response of the 1st mode resonance

Boron Nanowire: Length Determination

SEM images only give a two-dimensional projection of the cantilevered nanowire. It is critical to have accurate length measurement:

\[ E_b = \frac{64\pi^2 \rho}{\beta_n^4} \frac{L^4}{D^2} f_n^2 \]
\[ E_b \propto L^4 \]


A parallax method was used to reconstruct the correct three-dimensional representation of the nanowire based on two SEM images taken from different angles.

Schematic representation of a wire being partitioned into N segments, before (a) and after (b) rotation.
Boron Nanowire: Length Determination (con’t)

Top view and 45° tilted view of a nanowire

3-D reconstruction result

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Boron Nanowire: Oxide Layer

Without considering oxide layers:

\[ f_n = \frac{\beta_n^2}{2\pi} \frac{D}{L^2} \sqrt{\frac{E_{\text{Beam}}}{16\rho}} \]

\[ E_{\text{beam}} = \frac{64\pi^2 \rho_B}{\beta_n^4} \frac{L^4}{D^2} f_n^2 \]

Considering oxide layers:

\[ f_n = \frac{\beta_n^2}{2\pi} \frac{1}{L^2} \sqrt{\frac{EI}{\rho A}} = \frac{\beta_n^2}{2\pi} \frac{1}{L^2} \sqrt{\frac{E_B I_B + E_o I_o}{\rho A}} \]

\[ = \frac{\beta_n^2}{8\pi} \frac{1}{L^2} \sqrt{\frac{(E_B - E_o)(D-2T)^4 + E_o D^4}{(\rho_B - \rho_o)(D-2T)^2 + \rho_o D^2}} \]

Define: \( \alpha = (D-2T)/D \)

\[ E_B = \frac{1}{\alpha^2} \left( 1 + \frac{\rho_o}{\rho_B} \left( \frac{1}{\alpha^2} - 1 \right) \right) E_{\text{beam}} + E_o \left( 1 - \frac{1}{\alpha^4} \right) \]
A curved circular cross-section cantilevered beam can vibrate in two perpendicular directions: (1) in plane and (2) out of plane.
**FEA Analysis**

The simple beam theory is based on the assumption that the beam deflection is due to bending only and that transverse shear, rotatory inertia, and axial extension effects are negligible; for curved beams these assumptions are not correct.

Modal analysis was performed on several curved cantilever nanowires with ANSYS. The FEA model was based on the 3-D reconstruction of the nanowire configuration.

<table>
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<tr>
<th>Length (μm)</th>
<th>Diameter (nm)</th>
<th>Frequency (kHz)</th>
<th>Modulus assume straight (GPa)</th>
<th>Modulus FEA modeling (GPa)</th>
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<tbody>
<tr>
<td>16.5 ± 0.1</td>
<td>75 ± 2</td>
<td>346.4 (out)</td>
<td>198 ± 14</td>
<td>179.3</td>
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<td>378.7 (in)</td>
<td>237 ± 17</td>
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<td>7.9 ± 0.1</td>
<td>70 ± 2</td>
<td>1295 (in)</td>
<td>168 ± 19</td>
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<td>18.3 ± 0.1</td>
<td>116 ± 2</td>
<td>440.3 (in)</td>
<td>203 ± 14</td>
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<tr>
<td>25.5 ± 0.1</td>
<td>78 ± 2</td>
<td>102.1 (out)</td>
<td>91 ± 8</td>
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<td>105.9 (in)</td>
<td>98 ± 9</td>
<td>91.9</td>
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<td>7.8 ± 0.1</td>
<td>64 ± 2</td>
<td>1332.3 (in)</td>
<td>205 ± 25</td>
<td>194.2</td>
</tr>
</tbody>
</table>

**Summary**

1. Mechanical Resonance method based on simple beam theory, and some work has been done with FEA.

2. There are two commonly used methods to excite the mechanical resonance of cantilevered nanostructures: electrical excitation and mechanical excitation.

3. It is critical to have accurate geometry measurement.

4. Mechanical resonance method is an nondestructive and effective way to determine the elastic modulus of nanostructures; one aspect deserving careful scrutiny in the future is the low values for the modulus often obtained compared to the bulk material.