Nonlinear Vibration of Nano-scale Structures

Theory and Examples

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1. Introduction

Nonlinear vibration theory

To describe the vibration of mechanical structures, we often use linear algebraic equations. For most cases, this approach can explain the behavior of vibrating structures in the ideal case, but mechanical systems that we encounter experimentally are mostly nonlinear.

There are two typical characteristics of nonlinear behavior. First, the response of a nonlinear system is not linearly related to the input force or displacement. This concept can be expressed mathematically. In a nonlinear system, the following relation is not satisfied.

\[ f(ax) \neq af(x) \]

Second, nonlinear systems do not satisfy the principle of superposition. In linear systems, the response of \( a + b \) is same as the summation of the response \( a \) and \( b \) respectively. Mathematically, the principle of superposition is expressed below.

\[ f(a+b) = f(a) + f(b) \]

Generally speaking, nonlinear behavior occurs when displacement of vibration is beyond the range of small perturbation. In principle, nonlinearity can arise from any component of a vibrating system, i.e, mass, damper, or spring. In Fig. 1, a typical single degree of freedom vibration system is shown.

![Fig. 1 Single degree of freedom system](image-url)
The equation for the vibration of this system is expressed below.

\[ m \ddot{x} + c \dot{x} + k x = F \]

Nonlinearity arises when at least one coefficient of the above equation changes with respect to displacement, time, etc.

We will consider several nonlinear cases which occur frequently in experimental situations. One system we will discuss is the system which has variable mass and is expressed mathematically below.

\[ m(x) \ddot{x} + kx = 0 \]

We can see that the mass term in this equation is not a single value, but a function of the displacement \( x \). This case is common in systems such as that shown in Fig. 2. The colored area in the picture represents a tank filled with fluid. The vibrating body in the center is connected to a spring and faced with water on only one side. In this system, we expect that if the displacement is large, the effective mass will be changed, and nonlinear vibration will occur.

![Fig. 2 Variable mass system](image)

Another case of nonlinear vibration happens when frictional force changes. This case can be expressed mathematically in the following manner.

\[ m\ddot{x} + F(\dot{x}) + kx = 0 \]

We can see that frictional force is a function of velocity \( \dot{x} \). This kind of vibration usually occurs in the vibration of mechanical chatter. A schematic is shown in Fig. 3. In this figure we can see that the vibration of the mass is not only related to the restoring force of the spring, but also to the frictional force between the mass and the belt, which itself is related to the relative velocity between the mass and the belt.
The most frequent nonlinear case arises when the spring constant changes according to displacement. When the spring constant decreases with displacement, the system is referred to as a ‘softening spring’. A vibrating pendulum is a typical example of a softening spring and this system is shown in Fig. 4.

When the spring constant increases with displacement, the system is referred to as a ‘hardening spring’. A picture of a hardening spring is shown in Fig. 5.
The general form of the equation of motion for the above cases can be expressed mathematically in the form shown below, and this equation is known as the Duffing equation.

\[ \ddot{x} + c\dot{x} + \omega_0^2 x \pm \alpha x^3 = F \cos \omega t , \text{ where } \omega_0 = \sqrt{\frac{k}{m}} \]

The restoring force term in this equation can be extracted and expressed below.

\[ k \pm \frac{\alpha}{m} x^2 \]

While in a linear system the spring constant is simply expressed as \( k \), in this case the restoring force also includes a term related to \( x^2 \). The positive sign corresponds to the case where the spring constant becomes larger when displacement increases, so it is hardening spring case. The negative sign corresponds to the case where the spring constant becomes smaller when displacement increases, so it is softening spring case.

There is another frequent case of nonlinear vibration which follows the Mathieu equation given below.

\[ \ddot{y} + \mu \dot{y} + (a + \varepsilon \cos t)y = 0 \]

In this case, the parameter that is related to displacement changes with respect to time. The excitation force term which is extracted from the above equation can be changed as shown below.

\[ F = x(V_{dc} + V_{ac} \cos t) \]

As you can see, the excitation force in the Mathieu equation includes a constant term \( V_{dc} \) and a time varying term \( V_{ac} \). The time varying term causes nonlinear vibration in this system.
Characteristics of nonlinear vibration

There are several characteristics of nonlinear vibration, and they can be summarized as follows.

1) The frequency of vibration may be dependent on the amplitude of vibration.

2) While the frequency of excitation is increased, the amplitude of vibration may demonstrate a significant jump; the amplitude may suddenly fall after reaching a peak value, or suddenly jump upwards when the frequency of excitation is decreased.

3) When a system is excited by a harmonic force, the response will not only have the basic component but may consist of higher or lower harmonics.

4) The system can become self excited, and amplitudes of vibration may grow even without any external disturbance.

5) The system can become unstable under some conditions.

Most of the above characteristics of a nonlinear vibration system can be explained mathematically. The Duffing equation and Mathieu equations can explain the above characteristics with simple mathematics.
Nonlinear vibration (Duffing equation)

Definition of Duffing Equation

In the case of a vibrating pendulum, an equation of vibration can be derived as follows.

\[ ml^2 \ddot{\theta} + mgl \sin \theta = 0 \]

If we non-dimensionalize variables and replace the sine function with a linear approximation function, we can get a more simplified form.

\[ \dot{\theta} + \omega_0^2 (\theta - \frac{1}{6} \theta^3) = 0 \]

We can see a \( \theta^3 \) term in this case and as we already said, this term will cause nonlinear behavior.

The above equation frequently appears in vibration systems and is called the Duffing equation.

The more generalized form of the Duffing equation is provided below.

\[ \ddot{x} + c \dot{x} + \omega_0^2 x \pm \alpha x^3 = F \cos \omega t \]

Solution of the Duffing Equation

In this equation, if we set the response as \( x_1 = A \cos \omega t \), the variable \( A \) will have a relation to the other coefficients in the above equation.

\[ \left[ (\omega_0^2 - \omega^2) A \pm \frac{3}{4} \alpha A^3 \right]^2 + (c \omega A)^2 = F^2 \]

As we can see in above equation, the magnitude of response \( A \) is related to the excitation frequency, the nonlinear coefficient, the damping coefficient, and the excitation amplitude \( F \). In Fig. 6, numerical simulation results from the above equation are shown. The top graph demonstrates a ‘hardening spring’ case and the bottom graph represents a ‘softening spring’ case. The simulations are based on three parameters, \( \alpha = 5, c = 20, \omega_0 = 20 \).

In this figure, interesting characteristics of nonlinearity are shown. In the case of the hardening spring, a resonance peak curves to the right, and in the case of the softening spring, the resonance peak curves to the left.
We can also see a jump phenomenon. As shown in Fig. 7, as the excitation frequency goes up, there is sudden jump or drop in response and this pattern is similar in both the softening and hardening spring cases, but the direction of the jump and drop is different.

Fig. 6 Simulation result of Duffing equation

Fig. 7 Jump Phenomenon
Subharmonic and Superharmonic Vibration in Duffing Equation

As we discussed earlier, in a nonlinear system, when the system is excited by a harmonic force, the response will not only have the basic component but may also consist of higher or lower harmonics.

In subharmonic vibration, when the system is actuated with $\omega$, vibrations also occur at $\omega_n = \omega/n$. In contrast, in a superharmonic system, vibrations also occur at $\omega_n = n\omega$.

Let’s consider the 1/3rd subharmonic case. Below is a typical Duffing equation with a sinusoidal input force.

$$\ddot{x} + \omega_0^2 x + \alpha x^3 = F \cos 3\omega t$$

The response including the 1/3rd subharmonic can be expressed as follows:

$$x(t) = A \cos \omega t + C \cos 3\omega t$$

After some mathematical analysis, we can get a relation between parameters. The magnitude of responses $C$ and $A$ have a relation with the input force, driving frequency, and the system’s natural frequency ($F$, $\omega$, $\omega_0$).

$$C = -\frac{F}{8\omega^2}$$

$$\omega^6 - \omega_0^2 \omega^4 - \frac{3\alpha}{256} (64A^3\omega^4 - 8AF\omega^2 + 2F^2) = 0$$

In a similar way, the 3rd superharmonic can also be analyzed. Let’s consider the Duffing equation with a sinusoidal input force of a different frequency.

$$\ddot{x} + \omega_0^2 x + \alpha x^3 = F \cos \omega t$$

The response including the 3rd superharmonic can be expressed as follows:

$$x(t) = A \cos \omega t + C \cos 3\omega t$$

If we mathematically analyze the above equation we can get a relation among parameters $C$, $A$, $F$ and $\omega$.

$$C \approx \frac{-1}{4} \frac{\alpha A^3}{(\frac{3}{2} \alpha A^2 + \omega_0^2 - 9\omega^2)}$$

$$C \approx \frac{F - \omega_0^2 A + \omega^2 A - \frac{3}{4} \alpha A^3}{\frac{3}{4} \alpha A^2}$$
In Fig. 8, examples of subharmonic and superharmonic vibration are shown. Besides the original response, we can see additional responses at different frequencies.

\[
-A^5\left(\frac{15}{16}\alpha^2\right) + A^3\left(\frac{33}{4}\alpha\omega^2 - \frac{9}{4}\alpha\omega_0^2\right) + A^2\left(\frac{3}{2}\alpha F\right) \\
+ A(10\omega_0^2 - 9\omega^4 - \omega_0^4) + (\omega_0^2 F - 9\omega^2 F) = 0
\]

Fig. 8 Example of Subharmonic and Superharmonic
Nonlinear vibration (Mathieu equation)

The Mathieu equation occurs frequently in vibration systems and can be expressed in the form given below.

\[ \ddot{y} + \mu \dot{y} + (a + \varepsilon \cos t)y = 0 \]

If the system has negligible damping, the above equation can be more simply expressed.

\[ \ddot{y} + (a + \varepsilon \cos t)y = 0 \quad \ddot{y} = -(a + \varepsilon \cos t)y \]

The Mathieu equation arises when the excitation force has a time varying component (AC component) and a constant component (DC component).

The response of the above equation is stable for some cases and unstable for other cases. If system is stable, response will be sinusoidal with constant amplitude or decaying amplitude, and if system is unstable, amplitude of response will be increasing. Assuming that response is sinusoidal and amplitude of response is constant, we can get a relation between \( a \) and \( \varepsilon \), and this will be the boundary of stable and unstable region. The procedure for getting this relation is like this. If we use the perturbation method, the response can be expressed in the form shown below.

\[ y(t) = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \cdots \quad a = a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + \cdots \]

If we plug in the above relation into the equation of motion, we can obtain a series of relations.

\[ \varepsilon^0 : \dot{y}_0 + a_0y_0 = 0 \]
\[ \varepsilon^1 : \ddot{y}_1 + a_0\dot{y}_1 + a_1y_1 + y_0 \cos t = 0 \]
\[ \varepsilon^2 : \dddot{y}_2 + a_0\dddot{y}_2 + a_2y_2 + a_1\dot{y}_1 + y_1 \cos t = 0 \]

From the first equation, the response can be set as this

\[ y_0(t) = \begin{cases} \cos \sqrt{a_0}t = \cos \frac{n}{2}t \\ \sin \sqrt{a_0}t = \sin \frac{n}{2}t \end{cases} \]

When \( n = 0 \), if we plug in the above response to the other relations, we can obtain a relation between \( a \) and \( \varepsilon \) based on assumption that response is sinusoidal.

\[ \ddot{y}_1 + a_1 \cos t = 0 \quad a_1 = 0 \]
\[ y_1(t) = \cos t + \alpha \quad a_2 = -\frac{1}{2} \]
\[ \dddot{y}_2 + a_2 + (\cos t + \alpha) \cos t = 0 \]
In a similar way, we can get a relation between \( a \) and \( \varepsilon \) when \( n = 1, 2, 3, 4 \ldots \).

This is relation when \( n \) is 1.

\[
a = \frac{1}{4} \varepsilon - \frac{\varepsilon^2}{8} + \cdots \quad a = \frac{1}{4} \varepsilon + \frac{\varepsilon^2}{8} + \cdots
\]

This is relation when \( n \) is 2.

\[
a = 1 - \frac{1}{12} \varepsilon^2 + \cdots \quad a = 1 + \frac{5}{12} \varepsilon^2 + \cdots
\]

If we draw the above relations for \( n = 0,1,2 \), we get a stability diagram and this is shown in Fig. 9.

**Fig. 9** Stability diagram for the parametric resonance for \( n = 0-2 \)
Quality factor and effect of fluid

Even though system includes nonlinear characteristics, nonlinear vibration can be shown in structures which have high quality factor. Let’s discuss about it. If time scale signal of vibration is expressed as this,

\[ x(t) = x_0 e^{\sigma t} e^{-i \omega_0 t} \]

quality factor is defined as ratio between oscillating term and decaying term.

\[ Q = 0.5 \left| \frac{\omega_0}{\sigma} \right| \]

Two signals with different quality factor (10, 100) are shown in Fig. 10. In case of low quality factor signal it decays fast, and when quality factor is large signal decay very slow. If these signals transformed in frequency domain, we can check the sharpness of vibration peak is different at each case. The signal with high quality factor has sharper peak than low quality factor case. If we compare the result in Fig. 11 and simulation result in Fig. 6, we can realize that nonlinearity can happen when quality factor is large.

![Fig. 10 Two signals with different Q factor (10, 100)](image)

There is an alternative way to get quality factor directly from frequency response result. In this case, we can easily calculate quality factor by determining resonance frequency and its half power frequency, and equation for getting quality factor can be given like this.
Fig. 11 Fourier transformed result of two signals with different Q factor (10, 100)

\[ Q \approx \frac{1}{2\zeta} \approx \frac{\omega_n}{\omega_2 - \omega_1} \]

Graphical view of getting \( \omega_n \) and \( \Delta\omega = \omega_2 - \omega_1 \) is described in Fig. 12.

Fig. 12 An alternative way to get Q factor from frequency transformed signal
Quality factor changes with respect to its shape and material, but critically depends on damping of surrounding material. Fig. 13 shows that Q factor changes according to pressure of surrounding fluid. From this result, we can guess that nonlinear vibration can be shown more easily in vacuum condition.

Fig. 13 Calculated $Q$ factor as a function of pressure in dry nitrogen at 300 K
2. Realization of nonlinear vibration in nano-scale structures

General Overview

In previous chapters, general cases of nonlinear vibration have been discussed. The question now is what kinds of nonlinear vibration phenomenon appear in nano-scale structures.

Currently, several nonlinear vibration phenomena in nano-scale structures have been reported. There are two kinds of nonlinear vibration in nano-scale structures. The first case includes nonlinear systems that can be explained by the Duffing equation. This case typically arises in both ends fixed resonators. [3, 4] In this type of resonator, if displacement goes beyond the range of small perturbations, tension is applied to the resonating structure and this makes it stiffer. This situation is identical to the ‘hardening spring’.

The other nonlinear case is described by the Mathieu equation. This case typically occurs in one end fixed resonators,[5] but it has been reported that it can also arise in the both ends fixed case. The vibration, however, is flexural rather than torsional, so the basic principle is similar to the one end fixed case. [6] If a cantilever type one end fixed beam is actuated by an electric field, we can apply both an AC and a DC field at the same time, and this actuation method will induce the \(a\) and \(\varepsilon\) terms of Mathieu equation.

In Table 1, the current research status of nano-scale structure vibration is summarized. In some research, nonlinear vibration is reported, but all of the research can be described by either the Duffing equation or the Mathieu equation.

![Both ends fixed type and one end fixed type resonator](image)

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**Fig. 14 Both ends fixed type and one end fixed type resonator**
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<td></td>
<td>170 kHz</td>
<td>Length) 18.9µm, Dia) 110nm</td>
<td>Not shown</td>
</tr>
<tr>
<td></td>
<td>150 kHz</td>
<td>Length) ~20µm, Dia) 200nm</td>
<td>Not shown</td>
</tr>
<tr>
<td>Roukes group, Cleland group</td>
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<td>Length) 1.3µm, Dia) 43nm</td>
<td>Shown</td>
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<tr>
<td></td>
<td>3.8 MHz</td>
<td></td>
<td>Shown</td>
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<tr>
<td>Craighead group</td>
<td>9 MHz</td>
<td>Length) 7µm, Dia) 200nm</td>
<td>Shown</td>
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<tr>
<td>McEuen group</td>
<td>54 MHz</td>
<td>Length) 1.5µm, Dia) 1~4nm</td>
<td>Not shown</td>
</tr>
</tbody>
</table>
Nonlinear vibration (Duffing equation case #1)

Let’s look at the case of nonlinear vibration from the Craighead group.[3] This case is a nonlinear system which can be explained by the Duffing equation. The equation of motion is below.

\[
\ddot{\delta} + K_1 \dot{\delta} + K_3 \delta^3 + \frac{1}{\omega_0 Q} \dot{\delta} = \frac{F(\omega t)}{M}
\]

Nonlinear vibration is induced from the \(K_3\) term in the above equation and this term is due to stretching of the beam. Using the stress-strain relation this term can be expressed in the form below.

\[
K_3 = \frac{E_{ab}}{2ML^3}
\]

\(M\) is the mass of the paddle and the other parameters are shown in Fig. 11. As shown in Fig. 12, we can see nonlinear behavior clearly.

Fig. 15 SEM image of nanofabricated paddle oscillator

Fig. 16 Nonlinear vibration response of nanofabricated paddle oscillator
Nonlinear vibration (Duffing equation case #2)

Here is another case of nonlinear vibration in a both ends fixed structure. This research is from the Roukes group. They fabricated a Pt nanowire suspended structure as shown in Fig. 13 and vibrated it using magnetomotive detection. [4]

Differing from the case of the Craighead group, this structure does not have a paddle structure in the center, so it is actuated in high frequency. But it nonetheless exhibited similar nonlinear vibration characteristics. Jump and drop phenomenon also arose in this case and this is shown in the inset of Fig. 14.

Fig. 17 SEM image of suspended Pt nanowire device and measurement circuit

Fig. 18 Measured response of Pt nanowire device
Nonlinear vibration (Mathieu equation case #1)

Now we look at the case of nonlinear vibration from the Ruoff group.[5] This case is different from the previous two cases because the structure is of the one end fixed cantilever type and the nonlinearity is introduced by the method of excitation. Note that in the previous case the nonlinearity arose from changes in the spring constant of the structure. The equation of motion is provided below.

\[
\ddot{u}_i + \frac{c}{\gamma_i} \dot{u}_i + \frac{1}{\Omega^2} (\omega_i^2 + \frac{2q_iV_{dc}^2}{\rho A\gamma_i} + \frac{2\beta q_i V_{dc}^2}{\rho A\gamma_i} \cos t) u_i = 0
\]

This equation represents the damped case of the Mathieu equation, and parameters \(a\) and \(\varepsilon\) can be defined from the above equation.

\[
a = \Omega^2 \left[ \omega_i^2 + \frac{q_i V_{dc}^2}{\rho A\gamma_i} \right]
\]

\[
\varepsilon = \Omega^2 \frac{\beta q_i V_{dc}^2}{\rho A\gamma_i}
\]

A complex electric field of both DC and AC components was used to vibrate a boron nanowire using a nano manipulation stage inside a scanning electron microscope.

In Fig. 16 a stability diagram for this vibrating structure is shown, and the results are very close to those generated from analytical calculations.

![Fig. 19 Schematic of the experimental setup](image)
Fig. 20 (a) Stability diagram for parametric resonance for n=1
(b) Amplitude-frequency response curve
Nonlinear vibration (*Mathieu* equation case #2)

It has been reported that the *Mathieu* equation case occurs in another micro scale system. Let’s look at the case of nonlinear vibration from the Turner group.[6] As shown in Fig 17, the structure is of the both ends fixed type, however the vibration is not translational but torsional. By applying *AC* and *DC* fields with a comb actuator, this torsional vibrating system can be actuated. In principle, the actuation mechanism is the same as for the previous cantilever case but in this case they reported a stability diagram up to \( n=4 \) due to high stability and low damping of the structure. The equation of motion in this system is provided below.

\[
\ddot{\theta} + \frac{c}{\omega l} \dot{\theta} + \frac{1}{\omega^2 I} (k + \gamma A_{DC} + \gamma A_{AC} \cos \tau) \theta = 0
\]

In Fig. 18 experiment results are provided for four stability regions.

Fig. 21 SEM image of torsional oscillator

Fig. 22 Stability diagram for \( n = 1-4 \)
3. Other Cases of Vibration in Nano-scale Structures

Both ends fixed cases (Basic study)

There are a lot of interesting vibration phenomena in nano-scale structures. But not all of them exhibit nonlinear characteristics. Here we will discuss research results from the McEuen group.[7] They placed a SWCNT between two electrodes and vibrated it with electrostatic forces. Sensing the vibration motion was possible by measuring conductance changes of the SWCNT. Because the SWCNT is highly damped, nonlinearity does not show even though it is actuated in a high vacuum condition. Nonetheless, a resonance plot was obtained around 54MHz.

Fig. 23 Device geometry and diagram

Fig. 24 Measurement of the resonant response
**Both ends fixed cases (Application)**

Vibration of nano-structures can be applied to sensing gas molecules.[8] When gas molecules attach to structures, the mass of the structure will increase and this will reduce the resonance frequency. In this research, a Au/Pb structure was fabricated to sense gas molecules, and frequency reduction was observed when gas molecules attached. Results in Fig. 22 show that this Au/Pb structure is sensitive only to hydrogen gas.

![Fig. 25 SEM image of the device](image)

![Fig. 26 Frequency shift vs gas pressure](image)
One end fixed cases (Basic study)

Now we will discuss some interesting research results from the Min-Feng group.[9] They vibrated ZnO nanowires and measured the change of the Q factor in the air. Compared to vibration under vacuum conditions, when the nano-structures vibrate in air they do not show nonlinear characteristic because of the damping caused by the air. If we include the effect of damping from the air, the equation of vibration takes the form shown below.

\[
EI \frac{\partial^4 y(x,t)}{\partial x^4} + \rho_c A \frac{\partial^2 y(x,t)}{\partial t^2} = F_{\text{electro}} + F_{\text{hydro}}
\]

\[
F_{\text{electro}} = \frac{1}{2L} \frac{dC}{dz} \left( V_{dc} - \Delta \Phi + V_{ac} \sin \omega t \right)^2
\]

\[
F_{\text{hydro}} = -\alpha \mu \frac{\partial y(x,t)}{\partial t}
\]

Here the \( F_{\text{hydro}} \) term represents the effect of damping due to air.

Fig. 27 SEM image of a ZnO nanowire cantilever

Fig. 28 Q factor in air and liquid
**One end fixed cases (Application)**

The one end fixed cantilever is very sensitive to mass changes, so this system can be used to count the number of attached DNA molecules. These interesting research results are from the Craighead group.[10] Due to the added mass of the attached DNA molecules, the resonance frequency drops, and based on this frequency drop, the number of attached DNA molecules could be calculated.

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**Fig. 29 SEM image of a cantilever structure**

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**Fig. 30 Response of cantilever to DNA**
4. Conclusion

In this report nonlinear vibration theory in general mechanical systems is discussed and then specific nonlinear vibration phenomena that occur in nano-scale structures are also discussed. Nonlinear vibration phenomenon in nano-scale structure can be categorized into cases described by the Duffing equation or the Mathieu equation. Both cases exhibit interesting nonlinear vibration characteristics which cannot be seen in linear vibration. Other cases of general nonlinear vibration are also strongly encouraged in nano-scale structures although this has not been reported until recently.
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Appendices (Source code in MATLAB)

Response simulation of Duffing equation

clear all
w = 5:0.1:25;
A = 0:0.1:5;
alpha = 5;
w0 = 20;
c = 0.1;

for i=1:length(w)
    for j=1:length(A)
        F1(i,j) = sqrt(((w0^2-w(i).^2).*A(j)+3/4*alpha.*A(j)^3)^2+(c*w(i)*A(j))^2);
        F2(i,j) = sqrt(((w0^2-w(i).^2).*A(j)-3/4*alpha.*A(j)^3)^2+(c*w(i)*A(j))^2);
    end
end

ww = zeros(length(w),length(A));
for i=1:length(A)
    ww(:,i) = w';
end

AA = zeros(length(w),length(A));
for j=1:length(w)
    AA(j,:) = A;
end

subplot(211)
contour(ww,AA,F1,100)
xlabel('omega')
ylabel('abs(A)')
colorbar

subplot(212)
contour(ww,AA,F2,100)
xlabel('omega')
ylabel('abs(A)')
colorbar
Stability diagram of Mathieu equation

clear all
e   = -1:0.01:1;
a1  = -1/2*e.^2;
a21 = 1/4 - 1/2*e - 1/8*e.^2;
a22 = 1/4 + 1/2*e - 1/8*e.^2;
a31 = 1 - 1/12*e.^2;
a32 = 1 + 5/12*e.^2;

plot(a1,e,'.',a21,e,'.',a22,e,'.',a31,e,'.',a32,e,'.')
xlabel('a')
ylabel('eta')
Time scale and frequency transformed signals of different quality factor

clear all
sampling_freq=1000;
signal_length=5;
deltaT=1/sampling_freq;
t =0:1/sampling_freq:signal_length;
f = (0:(length(t)-1))/(length(t)*deltaT);

wi1= -1;
wr1= 20;
Q1 = 0.5*abs(wr1/wi1)
x1 = exp(wi1.*t).*exp(-i.*wr1.*t);

wi2= -0.1;
wr2= 20;
Q2 = 0.5*abs(wr2/wi2)
x2 = exp(wi2.*t).*exp(-i.*wr2.*t);

figure(1)
subplot(211)
plot(t,real(x1))
xlabel('time')
ylabel('Displacement')

subplot(212)
plot(t,real(x2))
xlabel('time')
ylabel('Displacement')

figure(2)
subplot(211)
plot(f,abs(fft(real(x1))))
xlabel('Frequency(Hz)')
ylabel('Displacement')
axis([0 10 0 2000])

subplot(212)
plot(f,abs(fft(real(x2))))
xlabel('Frequency(Hz)')
ylabel('Displacement')
axis([0 10 0 2000])